Theoretical and Numerical Approach to "Magic Angle" of Stone Skipping

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We investigate the condition for the bounce of circular disks which obliquely impacts on the fluid surface. An experiment [C. Clanet, F. Hersen, and L. Bocquet, Nature (London) **427**, 29 (2004)] revealed that there exists a "magic angle" of 20° between a disk's face and water surface in which the condition of the lowest impact speed necessary for a bounce is minimized. We perform a three-dimensional simulation of the disk-water impact by means of the smoothed particle hydrodynamics. Furthermore, we analyze the impact with a model of the ordinary differential equation (ODE). Our simulation is in good agreement with the experiment. The analysis with the ODE model gives us a theoretical insight into the "magic angle" of stone skipping.

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Problems with impacts and ricochets of solid bodies against water surface have received a considerable amount of attention [1–6]. In the early stage, the problem was of importance in naval engineering concerning the impacts of canon balls on sea surface [7]. A projectile ricochets off the water surface if some conditions are fulfilled. The condition may rely on initial impact velocity $\boldsymbol{v}_0$, angle of impact velocity with respect to water surface θ, and specific gravity of the object $\sigma = \rho'/\rho$, where ρ' and ρ are the mass density of the object and the fluid, respectively. Investigations then revealed that there exists a maximum angle of incidence $\theta_{\max}$ for impacts of spheres, above which the rebound does not occur [8]. Besides, it was empirically found that the $\theta_{\max}$ relates to the specific gravity of a sphere as $\theta_{\max} = 18/\sqrt{\sigma}$ if the impact velocity were sufficiently large. This relation was theoretically explained using a simple model of an ordinary differential equation (ODE) [8,9]. In military engineering today, the problem of water impacts may be not as important as that of a century ago, however, recently it gained renewed interest under the studies of locomotion of basilisk lizards [10] and stone skipping [11].

This study is motivated by an experimental study of stone skipping, a bounce of a stone against water surface, by Clanet *et al.* [12]. They investigated impacts of a circular disk (stone) on water surface focusing on a single impact process and found that an angle about $\phi \simeq 20°$ between the disk's face and the water surface would be the "magic angle" which minimizes the lowest velocity for a bounce $v_{\min}$. In this Letter, we theoretically and numerically study the oblique impact of disks and water surface. Our simulation successfully agrees with the experiment. Moreover, we apply an ODE model [13] to the disk-water impact and obtain an analytical form of the $v_{\min}$ and maximum angle $\theta_{\max}$ as a function of initial disk conditions.

To perform a numerical simulation of the disk-water impact, we solve the Navier-Stokes equation using the technique of smoothed particle hydrodynamics (SPH) [14,15]. The SPH method is based on a Lagrangian description of fluid. Flow is represented by a set of *fluid particles* which moves in accordance with an equation of motion derived from the Navier-Stokes equation. This method does not require a grid for computation and thus has an advantage to treat free surface motion. Several representations of the viscous term have been proposed for this method. We adopt an artificial viscous term [16] which is simple for computation and sufficiently examined with Couette flow [17]. We neglect surface tension. Velocity of sound of the fluid is put, at least, 25 times larger than the incident velocity of the disk.

We characterize a disk-water impact with four control parameters: angle of incidence θ, tilt angle ϕ, Reynolds number $\mathrm{Re} = v_0 r/\nu$, and Froude number $\mathrm{Fr} = v_0^2/(gr)$, where ν is the kinetic viscosity of water, r is the radius of the disk, and g is the gravitational acceleration. Other parameters, which are specific gravity σ (=2.7) and spin velocity of disk ω (= 65 [rotations/sec]), are fixed. Note that in the experiment Fr typically ranges from 4.0 to 200 and Re is of order 10^5. In Fig. 1, we show the snapshots of our simulation with $\mathrm{Re} = 10^3$ and $\mathrm{Fr} = 25$.

In the following discussion, we analyze the ODE model which was originally introduced by Birkoff *et al.* [13]. The model is based on the following assumptions: (i) Hydrodynamic pressure p acting from water is proportional to $(\boldsymbol{v} \cdot \boldsymbol{n})^2$, where $\boldsymbol{v}$ is the speed of the body and $\boldsymbol{n}$ is the unit vector to the surface of the disk. (ii) For the part of the surface facing air, there is no hydrodynamic force. (iii) During the whole process, deformation of water surface is negligible, and the boundary between immersed and nonimmersed areas is simply given as the cross section to a horizontal plane at water level. We notice that the first assumption is reasonable because of the high Reynolds number for typical cases of stone skipping [18].

Figure 2 is a schematic of disk-water impact. The ODE model is first applied to the present problem by Bocquet [11]. Here we consider a simplified version of the model neglecting the force perpendicular to $\boldsymbol{n}$, where $\boldsymbol{n}$ is a normal vector to the disk area. Since the net force $\boldsymbol{f}$ to the disk from water is proportional to the area S of the

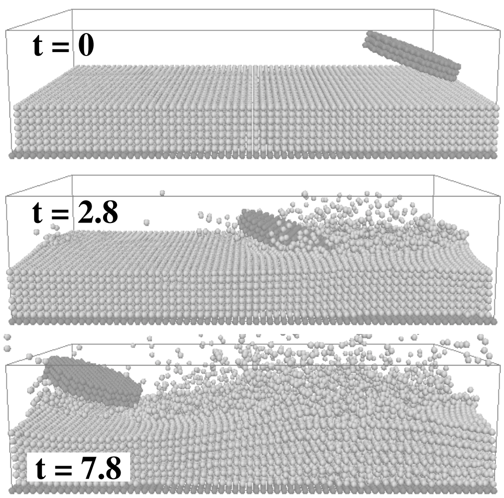

FIG. 1. Snapshots of the SPH simulation of a disk-water impact with $\phi = 20°$ and incident angle $\theta = 15°$. Specific gravity of the disk $\sigma = 2.7$. The unit of time is scaled with r/v_0. The number of fluid particles $N = 18\,837$.

water-disk interface, we have

$$\boldsymbol{f} = -\frac{1}{2} C_{\mathrm{D}} S \rho (\boldsymbol{v} \cdot \boldsymbol{n})^2 \boldsymbol{n}, \quad (1)$$

where ρ is the density of water. The drag coefficient C_{D} is not necessarily constant during the impact. For example, it varies according to the depth of immersion for vertical entries of spheres or disks [4,5]. Hence, in the present case, C_{D} would also depend on conditions such as initial angles θ, ϕ, and time. However, unfortunately, there is no experimental data available to determine C_{D} for our purpose. In this study, we use $C_{\mathrm{D}} \sim 1.4$, which is obtained from our SPH simulations of the typical initial angles ($\theta = 20°$, $\phi = 20°$) in the experiment ([12]), and assume that C_{D} is constant throughout the impact process.

For simplicity, we limit ourselves to the case that the angular velocity of the disk along the axis $\boldsymbol{n}$ is large enough, so that the angle ϕ remains constant during the process owing to a gyroscopic effect [11]. Both the experiments and our SPH simulations support the validity of this simplification.

Taking a frame of reference $O - \xi\zeta$ as shown in Fig. 2, we write the equation of motion as

$$\ddot{\xi} = -\frac{1}{\mathrm{Fr}} \sin\phi, \quad (2)$$

$$\ddot{\zeta} = \frac{C_D \lambda}{2\pi\sigma} \dot{\zeta}^2 S(z) - \frac{1}{\mathrm{Fr}} \cos\phi. \quad (3)$$

Here ξ, ζ, and z are the position of the lower edge of the disk in each coordinate, σ is the specific gravity, $\lambda = R/d$, and Fr is the Froude number. Immediately before the

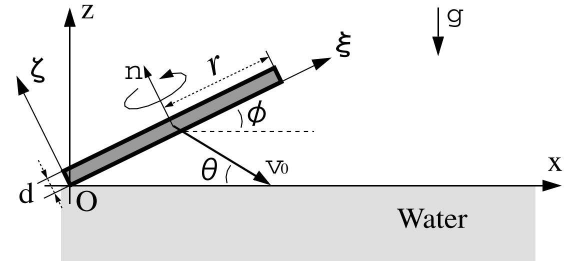

FIG. 2. An oblique water entry of a circular disk with incident angle θ and initial velocity $\boldsymbol{v}_0$. The edge of the disk is taken at the origin of the coordinates at $t = 0$. The radius and thickness of the disk is R and d. The angle between the disk and water plane is ϕ.

impact ($t = 0$), the position of the lower edge is set at the origin of the coordinate.

These equations give us straightforward insight into a necessary condition for stone skipping. Note that, to depart upward from the water surface, the velocity of the disk along with the ξ direction must be positive. Because the acceleration $\ddot{\xi}$ is negative constant, starting from the initial conditions with

$$\theta + \phi > \pi/2, \quad (4)$$

the initial velocity $\dot{\xi}(0) = \cos(\theta + \phi) < 0$ and $\dot{\xi}(t)$ would be negative all the time. Therefore the stone always dives into water and skipping of the stone would not take place. Our SPH simulation gives $\theta + \phi \lesssim 0.87\pi/2$ for the stone skipping domain in strong inertial limit ($F \to \infty$). We consider that this discrepancy comes from the drastic simplification of the ODE model; the force from water acts only on the front side of the disk. In SPH simulation, the force could act on the back side when cavity behind the disk is collapsed.

Let us consider the lowest velocity for a bounce $v_{\min}$ as a function of the angles θ and ϕ. A straightforward criterion whether a stone skips or not would be the vertical position after the entry into water surface; if the stone could recover the position higher than the water level, one could say it skips (criterion A). However, to make the analysis simple, we adopt an alternative criterion (criterion B); if the velocity $\dot{z}$ of the disk changes its sign to positive we regard that ricochet takes place. Under this definition, the entry velocity, such that the trajectory of the disk has an inflection point on its locally horizontal (parallel with x axis) line, would give the minimum velocity $v_{\min}$ for ricochet.

We can derive an equation which describes trajectories of disk motion. Equation (2) could be easily integrated with initial conditions $\xi(0) = 0$ and $\dot{\xi}(0) = \cos(\theta + \phi)$. Using the expression of $\xi(t)$, one could replace the time derivative in Eq. (3) with that of ξ and obtain

$$\left\{\cos^2(\theta + \phi) - \frac{2\sin\phi}{\mathrm{Fr}} \xi\right\} \zeta'' - \frac{\sin\phi}{\mathrm{Fr}} \zeta' = \frac{C_D \lambda}{2\pi\sigma} \left\{\cos^2(\theta + \phi) - \frac{2\sin\phi}{\mathrm{Fr}} \xi\right\} \zeta'^2 S(z) - \frac{1}{\mathrm{Fr}} \cos\phi, \quad (5)$$

where the prime indicates the derivative with respect to ξ.

Assume now that the disk entered into the water at the minimum velocity $v_{\rm min}$. Because the second derivative ζ'' is an invariant under the rotational transformation of coordinates, on the inflection point (ξ^*, ζ^*), $\zeta' = -\tan\phi$, and $\zeta'' = 0$. Thus we have

$$\frac{C_D\lambda}{2\pi\sigma}\left\{\cos^2(\theta+\phi) - \frac{2\sin\phi}{\mathrm{Fr}_{\rm min}}\xi^*\right\}S(z^*)\tan^2\phi - \frac{1}{\mathrm{Fr}_{\rm min}\cos\phi} = 0, \quad (6)$$

where $\mathrm{Fr}_{\rm min} = v_{\rm min}^2/gR$. In order for the criterion B to be satisfied, it is necessary that the inflection point exists in the domains of $\xi^* > 0$ and $z^* < 0$. It turns out that, in Eq. (6), ξ^* has the maximum value $\hat{\xi}^*$ when the disk is fully immersed, i.e., $S(z^*) = \pi$. Solving Eq. (6) for $F_{\rm min}(v_{\rm min})$, we finally obtain an expression for $v_{\rm min}$ as

$$v_{\rm min} = \frac{\sqrt{2gR}}{\cos(\theta+\phi)}\left\{\hat{\xi}^*\sin\phi + \frac{\sigma\cos\phi}{C_{\rm D}\lambda\sin^2\phi}\right\}^{1/2}. \quad (7)$$

We could derive the critical incident angle $\theta_{\rm cr}$ in the same way. Solving Eq. (6) for θ,

$$\theta_{\rm cr} = \arccos\sqrt{\frac{2}{\mathrm{Fr}}\left(\hat{\xi}^*\sin\phi + \frac{\sigma\cos\phi}{C_{\rm D}\lambda\sin^2\phi}\right)} - \phi. \quad (8)$$

Note that, in the limit $F \to \infty$, this equation again gives $\theta_{\rm cr} + \phi = \pi/2$.

A position of the inflection point $\hat{\xi}^*$ still remains unknown. We treat $\hat{\xi}^*$ as a fitting parameter, which should be determined so as to agree with experiments. However, we cannot make a direct comparison between Eq. (7) and the experimental data ([12]) because $v_{\rm min}$ and $\theta_{\rm max}$ are acquired under criterion A in the experiment. We thus fit Eq. (7) with the result of our SPH simulations performed under criterion B and evaluate $\hat{\xi}^* = 2.6$. Because of the nature of criterion B, these analytical expressions for $v_{\rm min}$ and $\theta_{\rm max}$ should give a lower and an upper limit of the stone skipping domain, respectively.

We need to mention the parameters chosen in the following discussions. For the SPH simulation, $\sigma = 2.7$, $\lambda = 4.0$, and the angular velocity of the disk $\omega = 65$[rounds/s]. The possible highest Reynolds number of our SPH simulation is of the order 10^3, whereas $\mathrm{Re} \sim 10^5$ in experiments. However, we consider that our simulation is still comparable with the experiment, because the simulation of a normal impact of a sphere and a liquid surface shows that the drag coefficient would be almost constant as far as $\mathrm{Re} > 10^2$. Note that this result is consistent with an experiment by Moghisi and Squire [4]. For the ODE model, we chose the same parameters as that of the experiment: $\lambda = 9.1$ and $\sigma = 2.7$ unless mentioned particularly.

Figure 3 shows the domains of stone skipping in (ϕ, v) and (θ, ϕ) planes. For the minimum velocity $v_{\rm min}$, the SPH simulation successfully agrees with the experiment, and the theoretical results under criterions A and B also show the qualitative agreement.

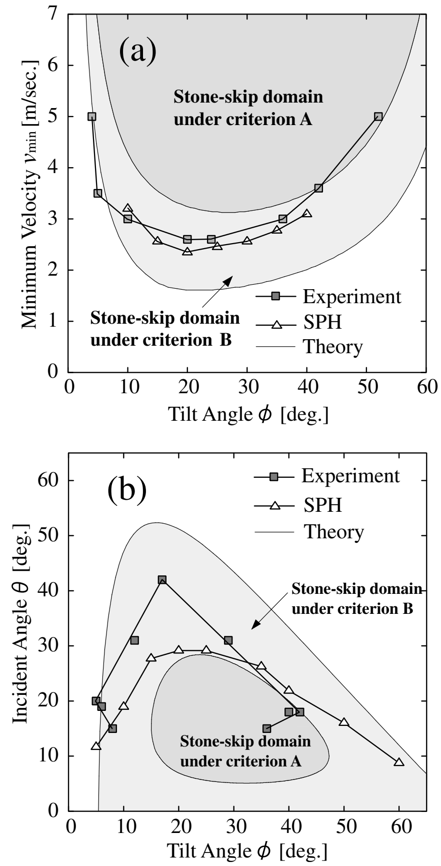

FIG. 3. A comparison of the stone skipping domains obtained from the experiment ([12]), SPH simulations, and our theory. (a) The stone skipping domain in the (v, ϕ) plane for a fixed $\theta = 20°$. (b) The stone skipping domain in the (θ, ϕ) plane for a fixed $v = 3.5$[m/s]. The boundary of the stone skipping domain under the criterion A in each graph are numerically drawn and those of B are the plot of Eqs. (7) and (8), respectively.

The experiment indicates that the stone skipping domain shrinks at $\theta < 20°$ in the (θ, ϕ) plane. The theoretical curve under criterion B does not reproduce this tendency, while that of criterion A shows similar behavior. This inconsistency is due to the assumption that the disk is fully immersed in the water when it reaches the inflection point. However, in the case that the θ is much smaller relative to the tilt angle ϕ, this is totally incorrect: only a small part of the disk is immersed during the impact process. The SPH simulation also shows different behavior with the experiment under $\theta < 20°$. We cannot present a clear explanation for this discrepancy. As for SPH simulations, the depth of immersion of the disk would be of the order of the fluid particle size of SPH at a very small incident angle. The numerical error hence becomes larger for small θ and for the domain $\theta < 10°$ simulation is not attainable.

The angle $\phi \simeq 20°$ is a characteristic for both (ϕ, v) and (θ, ϕ) planes in the experiment. Clanet *et al.* thus suggested that the angle $\phi_{\rm m} = 20°$ is the "magic angle" for

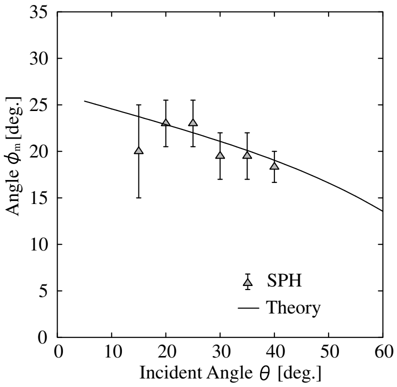

FIG. 4. Relation of incident angle θ and the angle ϕ_m. The SPH simulation is performed with $\sigma = 2.0$. The solid line is obtained numerically seeking the minimum of Eq. (7).

stone skipping. The angle should also be obtained solving equations $\partial v_{\min}/\partial\phi = 0$ or $\partial\theta_{\rm cr}/\partial\phi = 0$ with respect to ϕ. The former equation gives $\phi_{\rm m} = 22.9°$ and the latter gives $\phi_{\rm m} = 15.9°$, and these values depend on the incident angle as a matter of course. In Fig. 4, we show how the "magic angle" $\phi_{\rm m}$ is affected by the incident angle θ. Our theory suggests $\phi_{\rm m}$ decreases as the incident angle increases, and SPH simulation also shows a decreasing tendency. However, the change in $\phi_{\rm m}$ is sufficiently small: $\phi_{\rm m}$ changes only about 15% relative to the change of incident angle under $\theta = 40°$. We therefore conclude that the "magic angle" still remains around $\phi = 20°$ for the ordinal incident angle at stone skipping.

In summary, we have investigated an impact between a circular disk and a fluid surface based on two approaches: a three-dimensional numerical simulation using the SPH method and a simplified ODE model based on the equation of motion of the disk. The SPH simulation has qualitatively reproduced the experimental phase diagram for a successful bounce (stone skipping domain). The ODE model has given an analytical form of the lowest velocity $v_{\min}$ and the maximum angle $\theta_{\max}$. The analytical form of $v_{\min}$ implied that the "magic angle" $\phi_{\rm m}$ weakly depends on the angle of incidence θ. This tendency is also confirmed by the SPH simulation.

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